Abstract

In the present paper, we introduce new subclass $ST_{b}(b, \phi)$ of bi-univalent functions defined in the open disk. Furthermore, we find upper bounds for the second and third coefficients for functions in these new subclass using differential operator.

MSC: 30C45

Keywords: bi-univalent functions, coefficient estimates, starlike function, convex function, differential operator.

1 Introduction. Definitions And Preliminaries

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Further, by $S$ we shall denote the class of functions $f \in A$ which are univalent in $U$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $U$. However, the

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famous Koebe one-quarter theorem ensures that the image of the unit disk \( \mathbb{D} \) under every function \( f \in \mathcal{A} \) contains a disk of radius \( 1/4 \). Thus every univalent function \( f \) has an inverse \( f^{-1} \) satisfying \( f^{-1}(f(z)) = z \), \( (z \in \mathbb{D}) \) and \( f^{-1}(w) = w, (|w| < r_0(f), r_0(f) \geq \frac{1}{4}) \) where

\[
f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_3^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{1.2}
\]

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \mathbb{D} \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( \mathbb{D} \). We let \( \Sigma \) to denote the class of bi-univalent functions in \( \mathbb{D} \) given by (1.1). If \( f(z) \) is bi-univalent, it must be analytic in the boundary of the domain so that it can be continued across the boundary of the domain so that \( f^{-1}(z) \) is defined and analytic throughout \( |w| < 1 \).

Examples of functions in the class \( \Sigma \) are

\[ \frac{z}{1 - z}, -\log(1 - z), \] and so on.

The coefficient estimate problem for the class \( \mathcal{S} \), known as the Bieberbach conjecture, is settled by de-Branges [4], who proved that for a function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) in the class \( \mathcal{S} \), \( |a_n| \leq n \), for \( n = 2, 3, \ldots \), with equality only for the rotations of the Koebe function

\[ K_0(z) = \frac{z}{(1 - z)^2}. \]

In 1967, Lewin [9] introduced the class \( \Sigma \) of bi-univalent functions and showed that \( |a_2| < 1.51 \) for the functions belonging to \( \Sigma \). It was earlier believed that for \( f \in \Sigma \), the bound was \( |a_n| < 1 \) for every \( n \) and the extremal function in the class was \( \frac{1}{1 - z^2} \). E. Netanyahu [11] in 1969, ruined this conjecture by proving that in the set \( \Sigma \), \( \max_{f \in \Sigma} |a_2| \leq 4/3 \). In 1969, Suffridge [15] gave an example of \( f \in \Sigma \) for which \( a_2 = 4/3 \) and conjectured that \( |a_2| \leq 4/3 \). In 1981, Styer and Wright [14] disproved the conjecture that \( |a_2| > 4/3 \). Brannan and Clunie [2] conjectured that \( |a_2| \leq \sqrt{2} \). Kedziersawski [7] in 1985 proved this conjecture for a special case when the function \( f \) and \( f^{-1} \) are starlike functions. Brannan and Clunie [2] conjectured that \( |a_2| \leq \sqrt{2} \). Tan [16] in proved that \( |a_2| \leq 1.485 \) which is the best known estimate for functions in the class of bi-univalent functions.

Brannan and Taha [3] introduced certain subclasses of the bi-univalent function class \( \Sigma \) similar to the familiar subclasses \( S^*(\alpha) \) and \( C(\alpha) \) of the
univalent function class $\Sigma$. Recently, Ali et al.\[1\] extended the results of Brannan and Taha \[3\] by generalising their classes using subordination.

An analytic function $f$ is subordinate to an analytic function $g$, written $f(z) \prec g(z)$, provided there is a Schwarz function $w$ defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda \[10\], unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function $\phi$ with positive real part in the unit disk $U$, $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi$ maps $U$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, (B_1 > 0). \quad (1.3)$$

Recently Selvaraj and Karthikeyan \[8\] defined the following operator $D^m_{\alpha_1; \beta_1}(\alpha_1; \beta_1) f : \mathbb{U} \to \mathbb{U}$ by

$$D_0^0(\alpha_1; \beta_1) f(z) = f(z) \ast G_{\alpha_1, \beta_1}(\alpha_1, \beta_1; z),$$
$$D_1^1(\alpha_1; \beta_1) f(z) = (1 - \lambda)(f(z) \ast G_{\alpha_1, \beta_1}(\alpha_1, \beta_1; z)) + \lambda z (f(z) \ast G_{\alpha_1, \beta_1}(\alpha_1, \beta_1; z))^{'},$$
$$D^m_{\alpha_1; \beta_1} f(z) = D^1_\lambda(D^{m-1}_{\alpha_1; \beta_1} f(z)), \quad (1.4)$$

where $m \in \mathbb{N}_0$, $\lambda \geq 0$.

If $f \in \mathcal{A}$, then from (1.4) we may easily deduce that

$$D^m_{\alpha_1; \beta_1} f(z) = z + \sum_{n=2}^{\infty} \frac{[1 + (n - 1)\lambda]^m (\alpha_1)_{n-1} \cdots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_s)_{n-1}} \frac{a_n z^n}{(n - 1)!}. \quad (1.5)$$

Special cases of the operator $D^m_{\alpha_1; \beta_1} f$ includes various other linear operators which were considered in many earlier work on the subject of analytic and univalent functions. If we let $m = 0$ in $D^m_{\alpha_1; \beta_1} f$, we have

$$D^0_{\alpha_1; \beta_1} f(z) = H^1_{\alpha_1, \beta_1}(\alpha_1; \beta_1) f(z)$$

where $H^1_{\alpha_1, \beta_1}(\alpha_1; \beta_1)$ is Dziok-Srivastava operator for functions in $\mathcal{A}$ (see [6]) and for $q = 2, s = 1 \alpha_1 = \beta_1, \alpha_2 = 1$ and $\lambda = 1$, we get the operator introduced by Salagean([13]). It can be easily verified from the definition of (1.5),

$$z (D^m_{\alpha_1; \beta_1} f(z))' = (\alpha_1 + 1) D^m_{\alpha_1; \beta_1} f(z) - \alpha_1 D^m_{\alpha_1; \beta_1} f(z). \quad (1.6)$$
Definition 1.1 Let \( b \) be a non-zero complex number. A function \( f(z) \) given by (1.1) is said to be in the class \( ST_{\Sigma}(b, \phi) \) if the following conditions are satisfied:

\[
f \in \Sigma \quad \text{and} \quad 1 + \frac{1}{b} \left( \frac{D_{\lambda}^{m+1}(\alpha_1, \beta_1) f(z)}{D_{\lambda}^{m}(\alpha_1, \beta_1) f(z)} - 1 \right) < \phi(z), \quad z \in U
\]

(1.7)

and

\[
1 + \frac{1}{b} \left( \frac{D_{\lambda}^{m+1}(\alpha_1, \beta_1) g(w)}{D_{\lambda}^{m}(\alpha_1, \beta_1) g(w)} - 1 \right) < \phi(z), \quad z \in U
\]

(1.8)

where the function \( g \) is given by (1.2).

Definition 1.2 Let \( b \) be a non-zero complex number. A function \( f(z) \) given by (1.1) is said to be in the class \( ST_{\Sigma}(\alpha_1, \beta_1, b, \phi) \) if the following conditions are satisfied:

\[
f \in \Sigma \quad \text{and} \quad 1 + \frac{1}{b} \left( \frac{D_{\lambda}^{m}(\alpha_1 + 1, \beta_1) f(z)}{D_{\lambda}^{m}(\alpha_1, \beta_1) f(z)} - 1 \right) < \phi(z), \quad z \in U
\]

(1.9)

and

\[
1 + \frac{1}{b} \left( \frac{D_{\lambda}^{m}(\alpha_1 + 1, \beta_1) g(w)}{D_{\lambda}^{m}(\alpha_1, \beta_1) g(w)} - 1 \right) < \phi(w), \quad w \in U,
\]

(1.10)

where the function \( g \) is given by (1.2).

2 Coefficient estimates

Lemma 2.1 [12] If \( p \in \wp \), then \(|c_k| \leq 2\) for each \( k \), where \( \wp \) is the family of functions \( p \) analytic in \( U \) for which \( \text{Re} p(z) > 0 \), \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \) for \( z \in U \).

Theorem 2.2 Let the function \( f(z) \in A \) be given by (1.1). If \( f \in ST_{\Sigma}(b, \phi) \), then

\[
|a_2| \leq \frac{B_1 \sqrt{B_1} |b|}{\sqrt{(4(1 + 2\lambda)^m - (1 + \lambda)2^m) B_1^2 b \lambda + (B_1 - B_2) \lambda^2 (1 + \lambda)^{2m}}}
\]

(2.1)

and

\[
|a_3| \leq \frac{(B_1 + |B_2 - B_1|) |b|}{\lambda (4(1 + 2\lambda)^m - (1 + \lambda)^{2m})}.
\]
Proof. Since $f \in ST_{\Sigma} (b, \phi)$, there exists two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0 = s(0)$, such that

$$1 + \frac{1}{b} \left( \frac{D_{\lambda}^{m+1} (\alpha_1, \beta_1) f (z)}{D_\lambda^m (\alpha_1, \beta_1) f (z)} - 1 \right) = \phi (r(z))$$

and

$$1 + \frac{1}{b} \left( \frac{D_{\lambda}^{m+1} (\alpha_1, \beta_1) g (w)}{D_\lambda^m (\alpha_1, \beta_1) g (w)} - 1 \right) = \phi (s(z)).$$

It is also written as

$$1 + \frac{1}{b} \left( \frac{D_{\lambda}^{m+1} (\alpha_1, \beta_1) f (z) - D_\lambda^m (\alpha_1, \beta_1) f (z)}{D_\lambda^m (\alpha_1, \beta_1) f (z)} \right) = \phi (r(z)) \quad \text{and}$$

$$1 + \frac{1}{b} \left( \frac{D_{\lambda}^{m+1} (\alpha_1, \beta_1) g (w) - D_\lambda^m (\alpha_1, \beta_1) g (w)}{D_\lambda^m (\alpha_1, \beta_1) g (w)} \right) = \phi (s(z)).$$

Define the functions $p$ and $q$ by

$$p (z) = \frac{1 + r(z)}{1 - r(z)} = 1 + p_1 z + p_2 z^2 + \cdots \quad \text{and} \quad q (z) = \frac{1 + s(z)}{1 - s(z)} = 1 + q_1 z + q_2 z^2 + \cdots.$$  

Or equivalently,

$$r (z) = \frac{p(z)}{p(z) + 1} = \frac{1}{2} \left( p_1 z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \left( p_3 + \frac{p_1}{2} \left( p_1^2 - p_2 \right) - \frac{p_1 p_2}{2} \right) z^3 + \cdots \right)$$

and

$$s (z) = \frac{q(z)}{q(z) + 1} = \frac{1}{2} \left( q_1 z + \left( q_2 - \frac{q_1^2}{2} \right) z^2 + \left( q_3 + \frac{q_1}{2} \left( q_1^2 - q_2 \right) - \frac{q_1 q_2}{2} \right) z^3 + \cdots \right).$$

It is clear that $p$ and $q$ are analytic in $\mathbb{U}$ and $p(0) = 1 = q(0)$. Also $p$ and $q$ have positive real part in $\mathbb{U}$ and hence $|p_i| \leq 2$ and $|q_i| \leq 2$. In the view of (2.3), (2.4)and (2.5), clearly,
Using (2.5) and (2.6), one can easily verify that

\[
\phi \left( \frac{p(z) - 1}{p(z) + 1} \right) = 1 + \frac{B_1}{2} p_1 z + \left( \frac{B_1}{2} \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2 \right) z^2 + \cdots \tag{2.7}
\]

and

\[
\phi \left( \frac{q(w) - 1}{q(w) + 1} \right) = 1 + \frac{B_1}{2} q_1 w + \left( \frac{B_1}{2} \left( q_2 - \frac{q_1^2}{2} \right) + \frac{B_2 q_1^2}{4} \right) w^2 + \cdots . \tag{2.8}
\]

Since \( f \in \Sigma \) has the Maclaurin series given by (1.1), computation shows that its inverse \( g = f^{-1} \) has the expansion given by (1.2). It follows from (2.6), (2.7) and (2.8) that

\[
(1 + \lambda)^m a_2 = \frac{1}{2\lambda} B_1 p_1 b, \tag{2.9}
\]

\[
4\lambda (1 + 2\lambda)^m a_3 - \lambda (1 + \lambda)^{2m} a_2 = \frac{1}{2} b B_1 \left( p_2 - \frac{1}{2} p_1^2 \right) + \frac{1}{4} b B_2 p_1^2 \tag{2.10}
\]

and

\[
-(1 + \lambda)^m a_2 = \frac{1}{2\lambda} B_1 b q_1, \tag{2.11}
\]

\[
\lambda \left( 8\lambda (1 + 2\lambda)^m - (1 + \lambda)^{2m} \right) a_2^2 - 4\lambda (1 + 2\lambda)^m a_3 = \frac{1}{2} b B_1 \left( q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{4} b B_2 q_1^2 \tag{2.12}
\]

From (2.9) and (2.11), it follows that

\[
p_1 = -q_1. \tag{2.13}
\]

Now (2.10), (2.12) and (2.13) gives

\[
a_2^2 = \frac{B_1^3 b^2 (p_2 + q_2)}{4 \left[ \left( 4 (1 + 2\lambda)^m - (1 + \lambda)^{2m} \right) B_2^2 b |\lambda| + (B_1 - B_2) \lambda^2 (1 + \lambda)^{2m} \right]}. \tag{2.14}
\]

Using the fact that \(|p_2| \leq 2\) and \(|q_2| \leq 2\) gives the desired estimate on \(|a_2|\).
\[ |a_2| \leq \frac{B_1 \sqrt{B_1} |b|}{\sqrt{\left(4 (1 + 2\lambda)^m - (1 + \lambda)^{2m}\right) B_1^2 b \lambda + (B_1 - B_2) \lambda^2 (1 + \lambda)^{2m}}}. \]

From (2.10)-(2.12), gives
\[
a_3 = \frac{b B_1}{2} \left[ \frac{\left(4 (1 + 2\lambda)^m - (1 + \lambda)^{2m}\right) p_2 + (1 + \lambda)^{2m} q_2}{8 \lambda [4(1 + 2\lambda)^{2m} - (1 + \lambda)^{2m}(1 + 2\lambda)^m]} \right]
+ \frac{2(1 + 2\lambda)^m p_1^2 (B_2 - B_1) b}{8 \lambda [4(1 + 2\lambda)^{2m} - (1 + \lambda)^{2m}(1 + 2\lambda)^m]}
\]

Using the inequalities \(|p_1| \leq 2, |p_2| \leq 2\) and \(|q_2| \leq 2\) for functions with positive real part yields the desired estimation of \(|a_3|\).

For a choice of \(\phi(z) = \frac{1 + Az}{1 + Bz}\), \(-1 \leq B < A \leq 1\), we have the following corollary.

**Corollary 2.3** Let \(-1 \leq B < A \leq 1\). If \(f \in ST_\Sigma \left( b, \frac{1 + Az}{1 + Bz} \right)\), then
\[
|a_2| \leq \frac{|b| (A - B)}{\sqrt{\left(4 (1 + 2\lambda)^m - (1 + \lambda)^{2m}\right) (A - B) b \lambda + (1 + B) \lambda^2 (1 + \lambda)^{2m}}}
\]
and
\[
|a_3| \leq \frac{|A - B| (1 + |1 + B|) |b|}{\lambda \left(4 (1 + 2\lambda)^m - (1 + \lambda)^{2m}\right)}. \]

**Theorem 2.4** Let the function \(f(z) \in \mathcal{A}\) be given by (1.1). If \(ST_\Sigma (\alpha, \beta_1, b, \phi)\), then
\[
|a_2| \leq \frac{(\alpha_1 + 1) B_1 \sqrt{B_1} |b|}{\sqrt{\left(4 (1 + 2\lambda)^m - (1 + \lambda)^{2m}\right) B_1^2 b (\alpha_1 + 1) + (B_1 - B_2) (1 + \lambda)^{2m}} (2.15)}
\]
and
\[
|a_3| \leq \frac{(\alpha_1 + 1) (B_1 + |B_2 - B_1|) |b|}{\left(4 (1 + 2\lambda)^m - (1 + \lambda)^{2m}\right)}. \]
Proof. Since $ST_\Sigma (\alpha_1, \beta_1, b, \phi)$, there exists two analytic functions $r, s : \mathbb{U} \rightarrow \mathbb{U}$, with $r(0) = 0 = s(0)$, such that

$$1 + \frac{1}{b} \left( \frac{D^m_\lambda (\alpha_1 + 1, \beta_1) f(z)}{D^m_\lambda (\alpha_1, \beta_1) f(z)} - 1 \right) = \phi (r(z))$$ \hspace{1cm} (2.16)$$

and

$$1 + \frac{1}{b} \left( \frac{D^m_\lambda (\alpha_1 + 1, \beta_1) g(w)}{D^m_\lambda (\alpha_1, \beta_1) g(w)} - 1 \right) = \phi (s(z)).$$

Using (2.3), (2.4), (2.7) and (2.8), one can easily verified that

$$(1 + \lambda)^m a_2 = \frac{(\alpha_1 + 1)}{2} B_1 p_1 b,$$ \hspace{1cm} (2.17)$$

$$4 (1 + 2\lambda)^m a_3 - (1 + \lambda)^{2m} a_2^2 = (\alpha_1 + 1) \left[ \frac{1}{2} b B_1 \left( p_2 - \frac{1}{2} p_1^2 \right) + \frac{1}{4} b B_2 p_1^2 \right]$$ \hspace{1cm} (2.18)$$

and

$$-(1 + \lambda)^m a_2 = \frac{(\alpha_1 + 1)}{2} B_1 p_1 b,$$ \hspace{1cm} (2.19)$$

$$\left( 8 (1 + 2\lambda)^m - (1 + \lambda)^{2m} \right) a_2^2 - 4 (1 + 2\lambda)^m a_3 =$$

$$= (\alpha_1 + 1) \left[ \frac{1}{2} b B_1 \left( q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{4} b B_2 q_1^2 \right] .$$ \hspace{1cm} (2.20)$$

From (2.17) and (2.19), it follows that

$$p_1 = -q_1.$$ \hspace{1cm} (2.21)$$

Now (2.18), (2.20), (2.21) and using the fact that $|p_2| \leq 2$ and $|q_2| \leq 2$, we have

$$|a_2| \leq \frac{|\alpha_1 + 1| B_1 \sqrt{B_1} |b|}{\sqrt{\left( 4 (1 + 2\lambda)^m - (1 + \lambda)^{2m} \right) B_1^2 b (\alpha_1 + 1) + (B_1 - B_2) (1 + \lambda)^{2m}}} .$$

From (2.18)-(2.20), gives

$$|a_3| \leq \frac{|\alpha_1 + 1| (B_1 + |B_2 - B_1|) |b|}{\left( 4 (1 + 2\lambda)^m - (1 + \lambda)^{2m} \right) .}$$
References


[12] Pommerenke, Christian. Univalent functions. With a chapter on quadratic differentials by Gerd Jensen. Studia Mathemati-
ica/Mathematische Lehrbücher, Band XXV. Vandenhoeck Ruprecht, Göttingen, 1975. 376 pp


